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Monotone Operators and Nonlinear Equations

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1. INTRODUCTION

Consider the equation

$$L(u) = N(u). \tag{1}$$

Here, L is a linear operator and N is nonlinear. There are three standard ways of treating equations of this type. The fixed-point method, the variational approach, and the method of successive approximations. The first two are nonconstructive and the third is constructive.

The purpose of this paper is to add a fourth method based on monotone operators. This is a constructive method which depends upon the method of successive approximations. However, instead of the usual approach we use monotonicity which is derived from the monotone character of the linear operator.

In Section 2, we discuss monotone operators. In Section 3, we show how this yields monotonicity. In Section 4, we show how this yields convergence. In Section 5, we show, using a result of Dini, that this convergence is uniform. In Section 6, we give an application to nonlinear differential equations with initial values. In Section 7, we consider two-point boundary value problems. In Section 8, we consider systems. In Section 9, we give some results for the nonlinear heat equation. In Section 10, we give some results for the nonlinear potential equation.

Other equations may be treated by the same methods.

2. MONOTONE OPERATORS

We say that a linear operator is monotone if

$$L(u) \geqslant L(v) \tag{1}$$

implies

$$u \geqslant v. \tag{2}$$

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Sometimes, this inequality is reversed. It is made clear how the argument may be modified.

Let us assume that N is always positive and monotone increasing. Examples are the functions u^2 and e^u. In many cases all we need is the monotone increasing character.

These concepts can be generalized to the case of symmetric matrices with the ordering of symmetric matrices [1].

The functions u and v usually satisfy the same boundary conditions. It follows by virtue of the linearity that we may write

$$L(w) \geqslant 0 \qquad \text{implies} \quad w \geqslant 0, \tag{3}$$

where w is 0 on the boundary.

3. Monotonicity

Let us consider the following successive approximations

$$\begin{aligned} L(u) &= 0 \\ L(u_{n+1}) &= N(u_n). \end{aligned} \tag{1}$$

By virtue of the assumptions concerning L and N, we have monotonicity.

4. Convergence

In the previous section, we have derived a sequence which is monotone increasing. It will be convergent if we show that the sequence is uniformly bounded. There are two ways we can proceed.

In the first place, we can impose conditions on N which guarantee this boundedness. In the second place, we often are in the situation where a fixed-point or a variational argument establishes the existence of a solution. Often, the uniqueness of solution is readily established. The same argument as above shows that this solution is an upper bound. Hence, we have convergence. We give an example of this below.

It is worthwhile pointing out that in this case we have convergence wherever a solution exists.

5. Uniform Convergence

In the foregoing section, we showed that we had monotone convergence. Using Dini's theorem, in many cases we can improve this to uniform convergence.

6. Differential Equations with Initial Conditions

Let us consider

$$u'' + 3u' + 2u = u^2, \qquad u(0) = c_1, \quad u'(0) = c_2, \tag{1}$$

From the Poincaré–Lyapunov theorem [2], we know that a solution exists throughout the entire t-interval if $|c_1|$ and $|c_2|$ are sufficiently small. The linear operator is monotone. This follows from general results, but can readily be established by using factorization and the result for the first-order case repeatedly. In the first-order case this monotonicity is evident because we have an explicit solution.

7. Two-Point Boundary Conditions

Consider the equation

$$u'' - u = u^3 \qquad u(0) = c, \quad u(T) = 0. \tag{1}$$

The monotonicity of the linear operator now depends upon the associated Green function. Using a variational characterization we have given a simple proof of this result in [3]. The argument there readily extends to partial differential equations such as the potential equation, and the same argument shows the variation, diminishing character of the Green function [4]. The argument there also extends to various classes of nonlinear equations.

The foregoing equation is the Euler equation associated with the functional

$$\int_0^T (u'^2 + u^2 + u^4)\, dt. \tag{2}$$

A simple functional analysis argument shows the existence of the minimum. If in the equation above we have a plus sign instead of a minus sign, we have a characteristic value problem. The associated functional need not have a minimum if the length of the interval is too large and we cannot conclude the nonnegativity of the associated Green function.

8. Systems

Consider the vector differential equation

$$x' - Ax = g(x), \qquad x(0) = c. \tag{1}$$

We must now consider the matrix exponential e^{At}. As we know [5], a necessary and sufficient condition that all the elements of this matrix be nonnegative is

$$a_{ij} \geqslant 0, \qquad i \neq j. \tag{2}$$

We can extend the definition of the monotone operator given above easily. All we have to do is demand that the inequalities hold, component by component.

9. Nonlinear Heat Equation

Consider the equation

$$\begin{aligned} u_t - u_{xx} &= N(u), \\ u(x, 0) &= g(x), \qquad && 0 \leqslant x \leqslant 1 \\ u(0, t) &= u(1, t) = 0, \qquad && t > 0. \end{aligned} \tag{1}$$

We assume that $N(u)$ is a power series in u lacking constant and first-degree terms. There is an analog of the Poincaré–Lyapunov theorem for this equation. If we assume the $| g(x)|$ is sufficiently small, the solution exists for all positive t.

The linear operator is monotone. This may be established in many ways. One way is to use the difference equation

$$u(x, t + \Delta) = (ux - \Delta, t) + u(x + \Delta, t)/2 + \Delta f(t). \tag{2}$$

Here, we assume that $f(t)$ is nonnegative and that t takes only discrete values $0, \Delta, \ldots$. This difference relation makes the nonnegativity of u clear. Passing to the limit as $\Delta \to 0$, we obtain the linear operator.

10. The Nonlinear Potential Equation

Consider the equation

$$u_{xx} + u_{yy} = N(u). \tag{1}$$

The equation

$$u_{xx} + u_{yy} = e^u, \tag{2}$$

an important equation in conformal mapping, was considered in [6].

The linear operator is monotone. This may be established by using a variational expression as described above.

References

1. R. Bellman and R. Vasudevan, Quasi-linearization and the matrix Riccati equation, *J. Math. Anal. Appl.* **64** (1978), 106–113.
2. R. Bellman, "Methods of Nonlinear Analysis," Vol. I, Academic Press, New York, 1969.
3. R. Bellman, On the non-negativity of Green's functions, *Boll. Un. Mat. Ital.* 12 (1957), 411–413.
4. R. Bellman, On variation—Diminishing properties of Green's functions, *Boll. Un. Mat. Ital.* **16** (1961), 164–166.
5. R. Bellman, "Introduction to Matrix Analysis," McGraw-Hill, New York, 1960; 2nd ed., 1970.
6. R. Bellman and R. Kalaba, "Quasilinearization and Nonlinear Boundary-Value Problems," American Elsevier, New York, 1965.